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LETTER TO THE EDITOR

# Exact results for the topological charge of (1 + 1)-dimensional lattice gauge theory†

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**Abstract.** The exact results for the topological susceptibility and the mean value of the topological charge density in (1 + 1)-dimensional lattice gauge theory are obtained by means of the Hamiltonian method.

Topology of lattice gauge theory (LGT) is an important problem that has been noted in recent years. How to define the topological charge on a lattice is a difficult problem, because continuity in spacetime is lost on the lattice. Lüscher [1] answered this problem by giving a method for constructing transition functions of lattice gauge fields. He pointed out that in order to get a satisfactory definition of topological charge $Q$ for all lattice gauge fields the charge $Q$ must have the following properties.

(i) $Q$ is defined for all lattice gauge fields with periodic boundary condition (PBC), except for a set of fields which has zero measure in the functional integral.

(ii) $Q$ assumes integer values only.

(iii) $Q=\Sigma_n q(n)$, where the topological charge density $q(n)$ is a local function of the lattice gauge field and $n$ runs over lattice sites.

(iv) In the continuum limit $a\to 0$, $q(n)$ agrees with that of the continuum:

$$q(n)=\begin{cases}(a^4/16\pi^2)\,\mathrm{Tr}\,F_{\mu\nu}(an)\tilde{F}_{\mu\nu}(an) & \tilde{F}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} \qquad D=4\\ (a^2/4\pi)\varepsilon_{\mu\nu}F_{\mu\nu}(an) & \qquad D=2\end{cases}$$

where $a$ is the lattice spacing and $\varepsilon_{\mu\nu\alpha\beta}$ and $\varepsilon_{\mu\nu}$ are respectively 4D and 2D total antisymmetric tensors.

Panagiotakopoulos [2] discussed 2D U(1) LGT by means of the method in [1] and obtained the numerical results for the topological susceptibility $\chi_t=\langle Q^2\rangle/V$ ($V$ being the volume of the lattice). The numerical results for $\chi_t$ in 4D SU(2) and SU(3) LGT have been given by the DESY group in [3]. The purpose of the present letter is to study the topological charge of (1+1)-dimensional U(1) LGT with PBC by means of the Hamiltonian method for LGT. Thus the transition function for (1+1)-dimensional U(1) LGT is constructed by application of Lüscher's method. Exact results for the topological charge $Q$ and $\chi_t$ for (1+1)-dimensional U(1) LGT is obtained by use of

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the relation between the $\theta$ vacuum energy and the topological susceptibility given by [4]. The results are compared with those in [2].

We study $(1+1)$-dimensional U(1) LGT with PBC (period $= L$), i.e.

$$U_\mu(n) = U_\mu(n+mL) \qquad \forall m \in \mathbb{Z} \tag{1}$$

for each link variable $U_\mu(n) \in \mathrm{U}(1)$.

We divide the lattice into cells $c(n)$

$$c(n) = \{x \in \mathbb{R} \mid 0 \leqslant (x_\mu - n_\mu) \leqslant 1, \quad \mu = 0, 1\}. \tag{2}$$

The transition function $v_{n,\mu}(x)$ valued gauge group, which is a gauge transformation, is defined on the intersection of $c(n)$ and $c(n-\hat{\mu})$, $f(n,\mu) = c(n) \cap c(n-\hat{\mu})$. Transition functions must satisfy the cocycle condition at the intersection point $x$ of four cells, i.e.

$$v_{n-\hat{0},1}(y_0=1)\, v_{n,0}(y_1=0) = v_{n-\hat{1},0}(y_1=1)\, v_{n,1}(y_0=0) \tag{3}$$

where $y_\mu = x_\mu - n_\mu$, $\mu = 0, 1$. Besides, due to PBC,

$$v_{n,\mu}(x) = v_{n+mL,\mu}(x+mL). \tag{4}$$

We define a parallel transporter from each corner

$$x = n + \sum_{\mu=0}^{1} z_\mu \hat{\mu} \qquad z_\mu \in \{0, 1\} \tag{5}$$

of the cell $c(n)$ to $n$:

$$\begin{aligned} w^n(x) &= U_0(n)^{z_0} U_1(n+z_0\hat{0})^{z_1} \\ U_\mu(n)^0 &= 1 \qquad U_\mu(n) = U_\mu(n). \end{aligned} \tag{6}$$

The transition function can be defined from $w^n(x)$

$$v_{n,\mu}(x) = w^{n-\hat{\mu}}(x)\, w^n(x)^{-1}. \tag{7}$$

We can show from (7) that

$$v_{n,0}(y_1=0) = v_{n,0}(y_1=1) = U_0(n-\hat{0}) \tag{8}$$

$$\begin{aligned} v_{n,1}(y_0=0) &= U_1(n-\hat{1}) \\ v_{n,1}(y_0=1) &= U_0(n-\hat{1})\, U_1(n-\hat{1}+\hat{0})\, U_0(n)^{-1}. \end{aligned} \tag{9}$$

Extending (8) and (9) respectively to $f(n,0)$ and $f(n,1)$

$$v_{n,0}(y_1) = U_0(n-\hat{0}) \qquad \forall y_1 \in f(n,0) \tag{10}$$

$$v_{n,1}(y_0) = (U_1(n-\hat{1})^{-1} U_0(n-\hat{1})\, U_1(n-\hat{1}+\hat{0})\, U_0(n)^{-1})^{y_0}\, U_1(n-\hat{1}). \tag{11}$$

Equations (10) and (11) obviously satisfy (8) and (9). Furthermore, it is easy to show the cocycle condition (3) holds also. For convenience in the Hamiltonian formulation we take the temporal gauge, i.e. $U_0(n) = 1$. Then we obtain

$$\begin{aligned} v_{n,0}(x) &= 1 \\ v_{n,1}(y_0=1) &= U_1(n-\hat{1}+\hat{0}) = v_{n+\hat{0},1}(y_0=0) \end{aligned} \tag{12}$$

i.e. $v_{n,\mu}(x)$ is a continuous function of $y_0$.

The topological charge [2] can be defined from transition functions

$$Q = \frac{i}{2\pi} \sum_n \sum_{\mu\nu} \varepsilon_{\mu\nu} \int_{f(n,\mu)} \mathrm{d}x\, v_{n,\mu}^{-1} \partial_\nu v_{n,\mu}. \tag{13}$$

Let

$$U_\mu(n) = \exp(\mathrm{i}agA_\mu(n)). \tag{14}$$

Under the temporal gauge, we can obtain from (13)

$$Q = -\frac{1}{2\pi}\sum_n \int_{f(n,1)} \mathrm{d}t\, ag\dot{A}_1 = -\frac{ag}{2\pi}\sum_n \int \mathrm{d}t\, \dot{A}_1 \tag{15}$$

where $\dot{A} = \partial_t A_1$ and $n_x$ represents the space component of $n$. Integrating over $t$ gives

$$Q = -\frac{ag}{2\pi}\sum_{n_x}[A_1(n_x, L) - A_1(n_x, 0)]. \tag{16}$$

From (14) and the PBC, we have

$$\exp(\mathrm{i}agA_1(n_x, 0)) = \exp(\mathrm{i}agA_1(n_x, L)) \tag{17}$$

so the difference between $A_1(n_x, L)$ and $A_1(n_x, 0)$ must be an integer multiple of $2\pi/ag$. As a result of (16), $Q$ is strictly an integer for $(1+1)$-dimensional U(1) LGT.

In $(1+1)$-dimensional U(1) LGT, the Lagrangian is

$$L = \frac{a}{2}\sum_{n_x}\dot{A}_1^2. \tag{18}$$

Introducing the $\theta$ parameter, the action becomes

$$S = \int \mathrm{d}t\, L_\theta = \int \mathrm{d}t\, L + \theta Q. \tag{19}$$

Hence we have from (15) and (19)

$$L_\theta = \frac{a}{2}\sum_{n_x}\dot{A}_1^2 - \frac{\theta ag}{2\pi}\sum_{n_x}\dot{A}_1. \tag{20}$$

Let

$$\Pi'_\theta = \frac{\partial L_\theta}{\partial \dot{A}_1} = a\dot{A}_1 - \frac{\theta ag}{2\pi}. \tag{21}$$

Thus the Hamiltonian is

$$\begin{aligned} H_\theta = \Pi'_\theta \dot{A}_1 - L_\theta &= \frac{1}{2a}\sum_{n_x}\left(\Pi'_\theta + \frac{\theta ag}{2\pi}\right)^2 \\ &= \frac{g^2 a}{2}\sum_{n_x}\left(\Pi_\theta + \frac{\theta}{2\pi}\right)^2 \end{aligned} \tag{22}$$

where $\Pi_\theta = (1/ag)\Pi'_\theta$ (analogous to the electric field) is a generator of the U(1) group, and its eigenvalues are $0, \pm 1, \pm 2, \ldots$. The vacuum is the lowest eigenstate of $H$; it can be labelled by using the eigenvalue of $\Pi_\theta$

$$|0\rangle = \begin{cases} |0\rangle & -\pi < \theta < \pi \\ |-1\rangle & \pi < \theta < 3\pi \\ |+1\rangle & -3\pi < \theta < -\pi \\ \text{etc.} \end{cases} \tag{23}$$

For $m\pi < \theta < (m+1)\pi$, $m = 0, \pm 1, \pm 2, \ldots$, therefore, the $\theta$ vacuum energy is

$$E_\theta = \frac{1}{Na}\langle H_\theta\rangle_\theta = \frac{g^2}{2}\left(\frac{\theta}{2\pi} - l\right)^2 \qquad \text{when } m = 2l \tag{24}$$

and

$$E_\theta = \frac{1}{Na}\langle H_\theta\rangle_\theta = \frac{g^2}{2}\left(\frac{\theta}{2\pi} - l - 1\right)^2 \qquad \text{when} \quad m = 2l+1 \tag{25}$$

where $N$ is the number of space lattice sites and $l = 0, \pm 1, \pm 2, \ldots$. Thus we can see from (24) and (25) that the $\theta$ vacuum energy exhibits periodic structure, as shown in figure 1.

From [4] we know the relation between the topological susceptibility and the $\theta$ vacuum energy

$$\chi_t = \left.\frac{\partial^2 E_\theta}{\partial \theta^2}\right|_{\theta=0} \tag{26}$$

and from (24)-(26) we can obtain

$$\chi_t = 1/4\pi^2\beta a^2 \tag{27}$$

where $\beta = 1/g^2 a^2$. The relation between the mean value of the topological charge density and the $\theta$ angle is

$$\frac{\langle Q\rangle}{V} = \begin{cases} \dfrac{g^2}{2\pi}\left(\dfrac{\theta}{2\pi} - l\right) & 2l\pi < \theta < (2l+1)\pi \\ \dfrac{g^2}{2\pi}\left(\dfrac{\theta}{2\pi} - l - 1\right) & (2l+1)\pi < \theta < (2l+2)\pi \end{cases} \tag{28}$$

where $l = 0, \pm 1, \pm 2, \ldots$. The mean value of the topological charge density is a periodic function of $\theta$, as shown in figure 2.

In this letter we have derived exact results for the topological charge density by using the Hamiltonian method for (1+1)-dimensional U(1) LGT. Numerical results for the topological susceptibility have been given in [2] for the Euclidean lattice. In

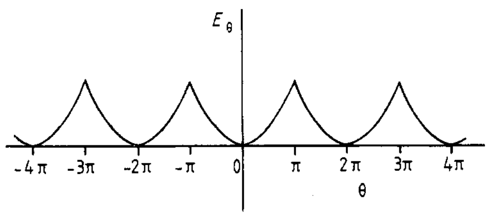


**Figure 1.** The periodic structure of the $\theta$ vacuum energy.

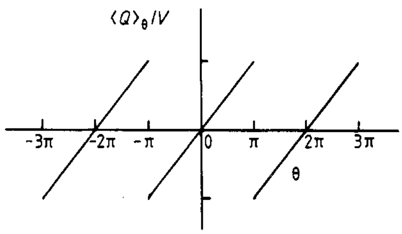


**Figure 2.** The relation between the mean value of the topological charge density and the $\theta$ angle.

our Hamiltonian formulation, the time direction is continuous while the space direction is discrete. Therefore, our results can only be compared with the numerical data in the scaling region. Our results correspond to the $\beta \to \infty$ limit. The tendency of the data given in [2] for large $\beta$ agrees with our result.

From equations (24), (25) and (28), we have

$$\langle Q \rangle_\theta / V = \partial E_\theta / \partial \theta \tag{29}$$

and

$$\lim_{\theta \to [(2l+1)\pi]^-} \frac{\partial E_\theta}{\partial \theta} = \frac{g^2}{4\pi} \tag{30}$$

$$\lim_{\theta \to [(2l+1)\pi]^+} \frac{\partial E_\theta}{\partial \theta} = -\frac{g^2}{4\pi} \tag{31}$$

i.e. $\partial E_\theta / \partial \theta$, and thereby the mean value of the topological charge density, is discontinuous at $\theta = (2l+1)\pi$ where the $\theta$ vacuum energy attains its maximum value. It is possible that there are physical effects.

The term $\theta / 2\pi$ in (22) can be considered as a background electric field. The $\theta$ parameter, which relates to the existing background electric field, corresponds to different values of the background field. There is a similar situation in the Schwinger model.

Consideration of the fermion effect and extension of the above analysis to the four-dimensional case remain to be investigated further.

*Note added.* After this work was completed, we noted a paper by Smit and Vink [5] in which the topological susceptibility for $(1+1)$-dimensional U(1) LGT has been obtained including scaling violation. Our results are exact in the Hamiltonian formulation and are valid even for finite $\beta$ in this formulation. Scaling violation will occur when the time axis is also discretised.

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